sphere. Ranger also showed that the drag coefficient based on initial diameter and for a wide range of conditions is about 3.0, a value in reasonable agreement with that of Rabin. In view of this, it is believed that the unusual C_D-Re curve shown in Schuyler's Fig. 1 is really a masked diameter or distortion effect and that C_D does not change that much. Accordingly, there appears to be no evidence to support a discernible effect of acceleration on C_D when the acceleration modulus is much less than unity. Selberg attributed his high values of C_D to surface roughness. This was further substantiated in a follow-on investigation by Sivier,[3] who used magnetically levitated sphere in a subsonic wind tunnel. Further, the results of Rudinger and Ingebo can not be taken as a case for an acceleration effect in that they used clouds of solid and liquid particles, respectively, with the attendant complex and unknown flowfields.

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Comment on "Solution of a Three-Dimensional Boundary-Layer Flow with Separation"

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In Ref. 1 a numerical solution is given for the equations of motion of three-dimensional laminar incompressible boundary layers. The method is restricted to non-negative tangential velocity components u and v. It is the purpose of this Comment to remove this restriction and show that numerical solutions can also be obtained for negative tangential velocity components.

For two-dimensional boundary layers, the stability of the difference equations can always be guaranteed by writing them implicitly for the direction normal to the wall. For three-dimensional boundary layers, the corresponding difference equations are only conditionally stable, even when they are written in an implicit form. This can be explained by the appearance of the term $v\partial/\partial y$ in the convective operator of the momentum equations (y denotes the second tangential co-ordinate). Consequently, there are several ways in which the difference quotients for the direction normal to the wall can be incorporated in the difference equations. In Ref. 1 $\Delta^2u/(\Delta z)^2, \Delta u/\Delta z$, etc. are formed at the four corner points of a mesh cell in the x-y plane. By determining the amplification factor[2] for the linearized differential equations, one can show that the resulting difference equations are stable for

$$0 \leq v\Delta x/u\Delta y \tag{1}$$

The validity of this condition is confirmed by the numerical results of Ref. 1. It seems worthwhile to point out that the difference quotients for the direction normal to the wall need only be formed at two diagonally opposite points of the mesh cell. The truncation error and the stability condition re-

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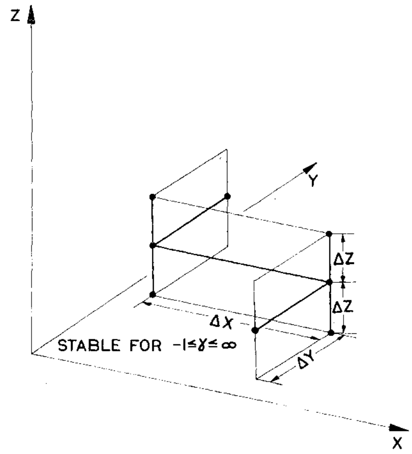

Fig. 1 Difference scheme for three-dimensional boundary layers.

main unchanged; however, the computational effort is very much reduced. It is understood that one of the two points in question must always be a point for which u and v are to be calculated.

Another way of combining the difference quotients[3] is indicated in Fig. 1. That difference scheme has the same truncation error as the one given in Ref. 1. Its amplification factor is

$$G(\Delta x) = \frac{2 + \gamma(1 - \cos k_2\Delta y) + \psi - i\gamma \sin k_2\Delta y}{2 + \gamma(1 - \cos k_2\Delta y) - \psi + i\gamma \sin k_2\Delta y} \tag{2}$$

The quantities γ and ψ stand for the following expressions:

$$\gamma = v\Delta x/u\Delta y \tag{3}$$

$$\psi = -\frac{2\mu\Delta x}{\rho u\Delta z^2}(1 - \cos k_1\Delta z) \tag{4}$$

For the difference equations to be stable, one must require that the absolute value of the amplification factor be less than or equal to unity.[2] This condition yields the following result:

$$-1 \leq v\Delta x/u\Delta y \tag{5}$$

If the increments Δx and Δy are taken to be positive, then u or v may become negative,† and the difference equations

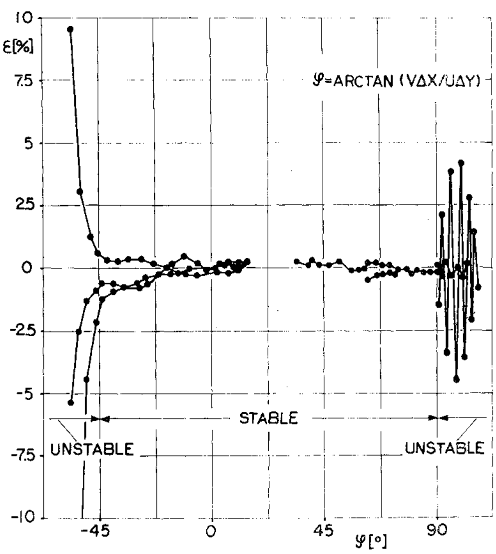

Fig. 2 Error of the shearing stress at the wall for the stable and unstable region of the difference scheme in Fig. 1.

† Negative tangential velocity components should not be confused with a time reversal in the heat conduction equation. The time reversal is equivalent to a negative conductivity.

are stable as long as condition (5) is satisfied. It is clear that condition (5) can be inversed by writing the amplification factor for Δy. Then all positive and negative values of u and v are, in principle, permissible.

Although the stability conditions given here were derived for the linearized equations of motion, the integration of the numerically exact difference equations shows that these conditions hold also true for the nonlinear problem. This can be seen in Fig. 2 taken from Ref. 4. Shown is the error of the shearing stress at the wall as a function of γ. As soon as the stability limits are exceeded, the error increases rapidly. There are other difference schemes that allow the tangential velocity components to be negative. These schemes are described in Refs. 3 and 5.

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Reply by Author to E. Krause

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IN his Comment E. Krause has shown that there exist stable solutions to the implicit finite-difference equations obtained from the three-dimensional boundary-layer equations. The existence of these solutions may extend the usefulness of the finite-difference techniques in three-dimensional boundary-layer theory; however, I think some further points should first be considered and investigated. The first point that should be investigated is the following. When boundary-layer equations are used to calculate a negative velocity in a boundary layer, it is implied that the previous flow history can be obtained just from the downstream-flow data. There are many examples of flows where this condition is violated, for example, more than one upstream condition will lead to the same downstream flow. Mathematically, this point becomes one of uniqueness. Dr. Krause has shown existence, but a nonuniqueness of the solution may be possible.

Another possible problem with the difference scheme proposed by Dr. Krause is in the upper regions of the boundary layer. In this flow region the viscous terms become small and the basic mathematical character of the equations could change. A good example of this is in supersonic boundary-layer theory. In the upper regions of the boundary layer, the equations could take on an almost hyperbolic character, and if the reversed flow finite-difference equations are used, the region of the influence of the characteristic nets will be violated. This situation could bring about a stability problem of a different character.

Therefore, I feel that the reversed flow difference equations proposed by Dr. Krause should be carefully studied and tested before they are given general acceptance. However, I do feel that they could be of great value, if the aforementioned points should prove to be nonvalid.

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Errata: "Buckling of Eccentrically Stiffened Cylinders under Torsion"

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IN Eq. (2c) the coefficients of the sixth derivatives (mixed or not) should be multiplied by $(\pi/L)^2$ and those of the fourth derivative by $(\pi/L)^4$. In the sixth line of Eq. (2c) λ_{xxst} should be multiplied by λ_{yy}. In the expression for α_i (see Appendix) the coefficient for the term $i^4\beta^2$ should read: $+ 2(\bar{e}_x\lambda_{xxst}\lambda_{yy} + \bar{e}_y\lambda_{yyr})$.

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